Research Report

Locating the Structural Zeros for Internal Anchor Tests: Including the Case of Rounded Formula Scores

Paul W. Holland

Research & Development

November 2005
RR-05-22

Locating the Structural Zeros for Internal Anchor Tests:
Including the Case of Rounded Formula Scores

Paul W. Holland
ETS, Princeton, NJ

November 2005

As part of its educational and social mission and in fulfilling the organization's nonprofit charter and bylaws, ETS has and continues to learn from and also to lead research that furthers educational and measurement research to advance quality and equity in education and assessment for all users of the organization's products and services.

ETS Research Reports provide preliminary and limited dissemination of ETS research prior to publication. To obtain a PDF or a print copy of a report, please visit:

http://www.ets.org/research/contact.html

Copyright © 2005 by Educational Testing Service. All rights reserved.

EDUCATIONAL TESTING SERVICE, ETS, and the ETS
logo are registered trademarks of Educational Testing Service.

Abstract

There are test-equating situations in which it may be appropriate to fit a loglinear or other type of probability model to the joint distribution of a total score on a test and a score on part of that test. For anchor test designs, this situation arises for internal anchor tests, which are embedded within the total test. Similarly, a part-whole relationship arises between two scores when a few test items are dropped from a test and a single group design is used to equate the scores of the full test to the part that remains after those items are deleted. In these part-whole situations, the resulting bivariate frequency distribution will exhibit *structural zeros* due to the fact that some scores on the total test are impossible for specific values on the partial test. Without knowing where the structural zeros are, it is impossible to distinguish them from zero frequencies that are simply due to size of the sample (i.e., sampling zeros). When probability models are estimated for these joint distributions, it is usual to require the models to assign positive probability to the sampling zeros but to avoid assigning positive probability to the structural zeros. To do this, it is important to be able to locate where the structural zeros are in the bivariate distribution. When the scores on the tests are consecutive integers, it is easy to determine the location of the structural zeros. This report gives a solution to the problem of locating the structural zeros that arise for a class of bivariate distributions that includes both number-right scores and formula scores that have been rounded to integer values. The result for rounded formula scores is a simple alteration of the case where all the scores are consecutive integers.

Key words: Test equating, probability models, discrete bivariate distributions, test scores

Acknowledgements

Dorothy T. Thayer, my longtime colleague and coauthor, reminded me that I had written notes on this subject in 1993, and she even produced a copy for me to read. I found them nearly incomprehensible, and so I wrote the present report in order to make the arguments clearer, at least to myself. I would like to thank her for her persistence in this matter and for her comments on an earlier draft of this report. In addition, I would like to thank Alina von Davier, Shelby Haberman, Dan Eignor, and Jim Ferris for their helpful suggestions on earlier versions of this report.

Table of Contents

Page

1. Introduction 1
2. Rounding Scores 3
3. Formula Scores 4
4. Structural Zeros for Rounded Scores 6
5. Three Simple Examples of Rounded Formula Scores 13
6. Implications for Degrees of Freedom for Chi-Square Tests 15
7. Summary 17

References 18

List of Tables

Page

Table 1. The Location of SZs for Example 1 14
Table 2. The Location of SZs for Example 2 14
Table 3. The Location of SZs for Example 3 15

List of Figures

Page

Figure 1. The standard parallelogram of non-SZs when **X** and **A** have only consecutive integer possible scores.. 2

Figure 2. The standard parallelogram and other areas relevant to SZs and non-SZs for rounded formula scores. 6

1. Introduction

This report is concerned with the location of impossible combinations of scores in certain types of bivariate score distributions. These cases routinely arise for the joint distributions of a total score, **X** + **A**, and one of its part scores, **A**. The part-whole relationship between the two scores and the limited range of the possible scores for **X** and **A** combine to make some scores for **X** + **A** impossible when **A** is fixed at a specific value. These impossible combinations of scores are called structural zeros (SZs) (Bishop, Fienberg, & Holland, 1975) and should be taken into account when models are fit to the joint distribution of the total score and the part score. In particular, these models should not put positive probability in the locations that are SZs. To avoid this, one must be able to locate where the impossible combinations occur in the distribution.

An important case of the part-whole relation arises with the nonequivalent groups with anchor test design when **A** denotes an internal anchor test raw-score and **X** + **A** denotes the total test raw-score. **A**nother situation where a part-whole relationship arises occurs when **A** contains a small number of items that are being deleted from the test and the shorter score, **X**, is to be equated to the longer test, **X** + **A**, using the single group design. In both of these situations, it is common to consider presmoothing the joint distributions using loglinear models as, for example, von Davier, Holland, and Thayer (2004) recommended when using the kernel method of test equating.

As discussed below, when **X** and **A** are number-right scored, it is easy to locate where the SZs are. However, in practice there may also be interest in the case where **X** + **A** and **A** are both formula scores that are rounded to integer values. This report gives a solution to this problem that includes rounded formula scores of a very general sort.

If **A** and **X** are both number-right scores, then both **A** and **X** + **A** can take on only consecutive integer values, starting at 0 and increasing up to $\mathbf{A}_{max}$, for **A**, and $\mathbf{X}_{max} + \mathbf{A}_{max}$ for **X** + **A**, where $\mathbf{A}_{max}$ and $\mathbf{X}_{max}$ are maximum values of **A** and **X**, respectively. Integer scores that have a nonzero lower bound can arise in practice when, for example, some items are flawed and, rather than being eliminated from the test, are scored as correct regardless of the answer given. In this situation, the smallest possible value of **X**, $\mathbf{X}_{min,}$ is greater than 0. Another situation where the lowest score is not 0 arises for test scores that include the score on an essay that is, for example, graded from 1 to 6.

Hence, it may happen in practice that **X** and **A** can only take on consecutive integer values, but not necessarily starting at 0. In this more general case, **A** can take on values from $\mathbf{A}_{min}$ to $\mathbf{A}_{max}$, and **X** + **A** can take on values from $\mathbf{X}_{min} + \mathbf{A}_{min}$ to $\mathbf{X}_{max} + \mathbf{A}_{max}$.

In the case just described, of consecutive integer scores, the location of the SZs is fairly easy to identify. It is easy to see that if **A** = j, then every value between $\mathbf{X}_{min}$ + j and $\mathbf{X}_{max}$ + j is a possible value of **X** + **A**, and no other values are. **A**ny value of **X** + **A** outside this range is impossible when **A** = j, so that the combinations of **A** = j and **X** + **A** scores outside this range are the SZs.

As j ranges from $\mathbf{A}_{min}$ to $\mathbf{A}_{max}$, the two-way array of frequencies for **X** + **A** by **A** has a *parallelogram of possible combinations* bounded by the cells indexed by

(i + $\mathbf{A}_{min}$, $\mathbf{A}_{min}$) for i = $\mathbf{X}_{min}$ to $\mathbf{X}_{max}$ (left-side boundary of the parallelogram),

(i + $\mathbf{A}_{max}$, $\mathbf{A}_{max}$) for i = $\mathbf{X}_{min}$ to $\mathbf{X}_{max}$ (right-side boundary of the parallelogram),

($\mathbf{X}_{min}$ + j, j) for j = $\mathbf{A}_{min}$ to $\mathbf{A}_{max}$ (top boundary of the parallelogram),

and

($\mathbf{X}_{max}$ + j, j) for j = $\mathbf{A}_{min}$ to $\mathbf{A}_{max}$ (bottom boundary of the parallelogram).

For the case of consecutive integer values for both **X** and **A,** the diagram in Figure 1 is a schematic representation of the SZs and non-SZs in the joint distribution of **X** + **A** and **A**.

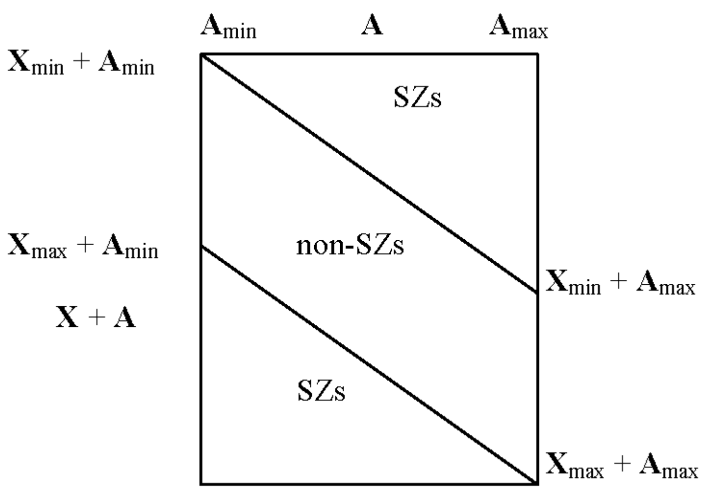

***Figure 1.* The standard parallelogram of non-SZs when X and A have only consecutive integer possible scores.**

The region described above and in Figure 1 is the *standard parallelogram* of non-SZs. In summary then, when the possible scores of **X** and **A** are both consecutive integers (including number-right scores), the standard parallelogram locates where the non-SZs are in the joint distribution of **X** + **A** and **A**, and all other locations in the joint distribution are the SZs.

2. Rounding Scores

In practice, more complicated cases than those discussed in section 1 may also arise. For example, both **A** and **X** need not have nonnegative integer-valued scores, but may be negative or have fractional parts. These types of scores arise when corrections for guessing are used, such as the well-known formula, R – (1/4)W, for scores from five-option multiple-choice (MC) tests. They can arise in other ways, such as theta estimates" from item response theory (IRT) models. This report does not make strong assumptions about the nature of these possibly negative and fractional scores at this. The primary motivation for this report is to extend the results described in Figure 1 to the case of rounded formula scores.

When fractional scores are present, the scores of **X** + **A** and **A** are often rounded separately to integer values. I do not know how widespread this practice is, but it certainly arises at ETS.

To denote the rounding method focused on in this report, the following rounding function, [x], for any real number x, is defined by

$$[x] = n \text{ if and only if } n - 0.5 \le x < n + 0.5, \text{ and } n \text{ is an integer.} \tag{1}$$

The rounding function in (1) corresponds to what is often called *rounding up* because numbers with fractional parts equal to 0.5 are rounded up to the nearest integer. This report uses this definition because it is the one traditionally used at ETS under the rubric of rounding in favor of the candidate. Of course, there are other ways to round numbers—such as rounding down or rounding even—but they play no role in this discussion.

The *remainder function*, r(x) is defined as follows

$$r(x) = x - [x], \tag{2}$$

so that for any real number, x,

$$-0.5 \le r(x) < 0.5, \tag{3}$$

and,

$$x = [x] + r(x). \quad (4)$$

The two functions [x] and r(x) have four useful properties, summarized in Lemma 1.

Lemma 1: For any integer n and real numbers x and y,

(a) $[n + x] = n + [x]$,

(b) $[r(x)] = 0$,

(c) if $x < y$ then $[x] \leq [y]$,

and

(d) $[x + y] = [x] + [y] + [r(x) + r(y)]$.

Proof: The first three properties are obvious. Property (d) follows from property (a) and the fact that $[x + y] = [[x] + r(x) + [y] + r(y)]$. QED.

The four properties in Lemma 1 simplify the discussion used in this report.

3. Formula Scores

This report uses the following framework for discussing formula scores that arise from corrections for guessing. **A** score, **S**, is a *formula score* here if it is of the form

$$\mathbf{S} = R_0 + \{R_1 - W_1\} + \{R_2 - (1/2)W_2\} + \{R_3 - (1/3)W_3\} + \ldots. \quad (5)$$

In (5), R_j denotes the number right and W_j the number wrong for items that are corrected for guessing by using a formula score of the form $R_j - (1/j)W_j$. Usually these are MC test items with j + 1 options or possible responses.

In (5), the sequence continues so that it includes all of the types of formula scores that are used in the test. $R_3 - (1/3)W_3$ is the formula score for four-option test items, and the sequence does not need to stop there if there are other types of test items on the whole test with more answer options. R_0 denotes a consecutive integer score for an item, such as an essay, which is graded from, say,1 to 6, or with any other consecutive integer values, say u to v. R_0 is also meant to include the *sum* of such item scores, the only proviso being that the possible values of R_0 are consecutive integers with no gaps.

Thus, the term *formula scores* includes scores that are the sums of scores that are corrected for guessing in possibly more than one way—that is, tests that include both four- and five-choice items—as well as other consecutive integer scores that are added to the sums of formula scores. This generality is needed to include the many special cases that arise at ETS. Remarkably, such generality does not interfere with the analysis.

One type of score that (5) does not cover arises when two different formula scores of the type described in (5) are each multiplied by a weight and then added together. Certain types of composite scores are like this, for example, R + 3.2T, where R is a number-right score for a set of multiple-choice questions and T is an essay score. These types of scores are not included in this analysis and must be dealt with in a different way.

There are some useful properties of rounded formula scores of the form defined by (5). To study them, this report uses Lemma 1a to write [**S**] as

$$[\mathbf{S}] = R_0 + R_1 - W_1 + R_2 + R_3 + \ldots + [-(1/2)W_2 - (1/3)W_3 \ldots]. \tag{6}$$

From (6) observe that the fractional parts of **S** all stem from the items that are answered incorrectly. Lemma 2 states, without proof, two simple but important observations about rounded formula scores of the form given in (5).

Lemma 2: If **S** is a formula score as defined in (5), then

(a) every integer value from [$\mathbf{S}_{min}$] to [$\mathbf{S}_{max}$] can be achieved by an integer value of **S**, except possibly for [$\mathbf{S}_{min}$], and,

(b) if [**S**] = [$\mathbf{S}_{max}$], then **S** = $\mathbf{S}_{max}$, an integer.

In part (a) in Lemma 2, it is possible that only an **S**-value with a fractional part can round to [$\mathbf{S}_{min}$]. **A**s a simple example, if **S** is from a three-item test and R – (1/4)W is the rule is used for scoring, then $\mathbf{S}_{min}$ = -3/4 and [$\mathbf{S}_{min}$] = -1. In this case, no achievable *integer* value for **S** can round to -1. Part (b) of Lemma 2 means that the only way to get the highest score on a formula-scored test is to get every item correct, an integer score. Nothing else rounds to the top score. Section 4 uses the two properties in Lemma 2 repeatedly.

4. Structural Zeros for Rounded Scores

Consider the joint occurrence of [**X** + **A**] = i and [**A**] = j. Then, just as in the consecutive integer case discussed in section 1, it is easy to see that some combinations of i and j are impossible and, therefore, create SZs.

For example, if **X** is a 10-item test, **A** is a 20-item test, and the scores are all of the form R – (1/4)W, then it is impossible for [**A**] = 20 and, at the same time, for [**X** + **A**] = 31 or higher or for [**X** + **A**] = 17 or less. [**X** + **A**] = 30 can only arise when **X** + **A** = 10 + 20, and [**X** + **A**] = 18 arises when **X** = -10/4 = – 2.5 and **X** + **A** = – 2.5 + 20 = 17.5, which rounds to 18. Note that the only way for [**A**] = 20 is for **A** = $\mathbf{A}_{max}$ = 20, as indicated in Lemma 2b.

Now suppose in the general case of rounded formula scores that [**A**] = j. What are the possible values of [**X** + **A**]? The answer is, at most, a slight alteration to the standard parallelogram described in Figure 1, but to state the final result (which the end of this section does), it is necessary to examine several situations. Figure 2 gives the schematic representation of the areas of the joint distribution of [**X** + **A**] and [**A**], which arises in the rest of the discussion.

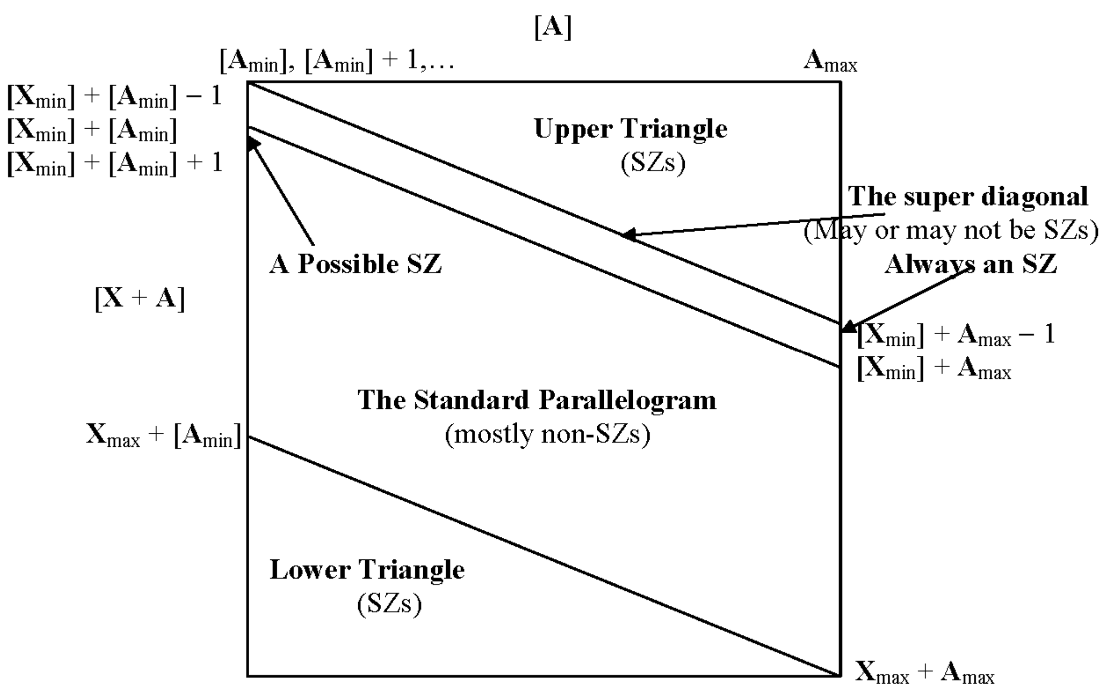

Figure 2. **The standard parallelogram and other areas relevant to SZs and non-SZs for rounded formula scores.**

Section 5 gives three simple examples that are intended to help the reader understand the more general derivations that follow it.

In general, Lemma 1c can be used to assert that $[\mathbf{X} + \mathbf{A}]$ must be within these limits,

$$[\mathbf{X}_{min} + \mathbf{A}] \leq [\mathbf{X} + \mathbf{A}] \leq [\mathbf{X}_{max} + \mathbf{A}] \ . \tag{7}$$

To compute various rounded sums of formula scores, this report makes repeated use of Lemma 1d, that is, the formula

$$[\mathbf{X} + \mathbf{A}] = [\mathbf{X}] + [\mathbf{A}] + [r(\mathbf{X}) + r(\mathbf{A})]. \tag{8}$$

In (7), the quantity $[\mathbf{X}_{max} + \mathbf{A}]$ has an upper bound that is easy to establish when $\mathbf{X}$ is a formula score. The result is summarized in Theorem 1.

Theorem 1: If $\mathbf{X}$ is a formula score and $[\mathbf{A}] = j$, then

$$[\mathbf{X} + \mathbf{A}] \leq \mathbf{X}_{max} + j. \tag{9}$$

Proof: From (7), examine $[\mathbf{X}_{max} + \mathbf{A}]$. But $\mathbf{X}_{max}$ is an integer, and from Lemma 1a $[\mathbf{X}_{max} + \mathbf{A}] = \mathbf{X}_{max} + j$. QED.

Theorem 1 shows that if $[\mathbf{A}] = j$ for any j from $[\mathbf{A}_{min}]$ to $[\mathbf{A}_{max}]$, then any $[\mathbf{X} + \mathbf{A}]$-score higher than $\mathbf{X}_{max} + j$ is impossible and produces an SZ. Thus, the lower triangle shown in Figure 2 contains only SZs.

Next, consider the insides of the standard parallelogram except for the left-side boundary. Theorem 2 gives the result.

Theorem 2: If $\mathbf{A}$ is a formula score and $[\mathbf{A}] = j > [\mathbf{A}_{min}]$ then every integer score from $[\mathbf{X}_{min}] + j$ to $[\mathbf{X}_{max}] + j$ can be achieved by $[\mathbf{X} + \mathbf{A}]$.

Proof: If $j > [\mathbf{A}_{min}]$ then, from Lemma 2a, $[\mathbf{A}] = j$ can be achieved by an integer value for **A.** Hence, assume that $r(\mathbf{A}) = 0$. But (8) then results in

$$[\mathbf{X} + \mathbf{A}] = [\mathbf{X}] + [\mathbf{A}] + [r(\mathbf{X}) + r(\mathbf{A})] = [\mathbf{X}] + j + [r(\mathbf{X}) + 0]$$

$$= [\mathbf{X}] + j + 0.$$

Now let $[\mathbf{X}]$ range from $[\mathbf{X}_{min}]$ to $[\mathbf{X}_{max}]$. QED.

Theorem 2 shows that if $[\mathbf{A}] = j > [\mathbf{A}_{min}]$, then no SZs can occur for $[\mathbf{X} + \mathbf{A}]$ from $[\mathbf{X}_{min}] + j$ to $[\mathbf{X}_{max}] + j$. This shows that, for all of the columns to the right of $[\mathbf{A}] = [\mathbf{A}_{min}]$ in Figure 2, the standard parallelogram includes only non-SZs for the joint distribution of $[\mathbf{X} + \mathbf{A}]$ and $[\mathbf{A}]$.

Next, consider the left-side boundary of the standard parallelogram in Figure 2. Theorem 3 gives the result.

Theorem 3: If $\mathbf{X}$ is a formula score and $[\mathbf{A}] = [\mathbf{A}_{min}]$, then every integer score from $[\mathbf{X}_{min}] + 1 + [\mathbf{A}_{min}]$ to $[\mathbf{X}_{max}] + [\mathbf{A}_{min}]$ can be achieved by $[\mathbf{X} + \mathbf{A}]$.

Proof: Use (8) to show that

$$[\mathbf{X} + \mathbf{A}] = [\mathbf{X}] + [\mathbf{A}] + [r(\mathbf{X}) + r(\mathbf{A})] = [\mathbf{X}] + [\mathbf{A}_{min}] + [r(\mathbf{X}) + r(\mathbf{A})] \tag{10}$$

If $[\mathbf{X}] \geq [\mathbf{X}_{min}] + 1$, then from Lemma 2 $[\mathbf{X}]$ is achievable by taking $\mathbf{X}$ to be an integer so that $r(\mathbf{X}) = 0$, and hence, for any $[\mathbf{X}] \geq [\mathbf{X}_{min}] + 1$, (10) becomes

$$[\mathbf{X} + \mathbf{A}] = [\mathbf{X}] + [\mathbf{A}_{min}] + [0 + r(\mathbf{A})] = [\mathbf{X}] + [\mathbf{A}_{min}]. \tag{11}$$

This shows that every integer score from $[\mathbf{X}_{min}] + 1 + [\mathbf{A}_{min}]$ to $[\mathbf{X}_{max}] + [\mathbf{A}_{min}]$ can be achieved by $[\mathbf{X} + \mathbf{A}]$. QED.

Theorem 3 shows that the left-side boundary of the standard parallelogram also contains non-SZs for the joint distribution of $[\mathbf{X} + \mathbf{A}]$ and $[\mathbf{A}]$, except for possibly the cell defined by $[\mathbf{X} + \mathbf{A}] = [\mathbf{X}_{min}] + [\mathbf{A}_{min}]$ and $[\mathbf{A}] = [\mathbf{A}_{min}]$. Theorem 4, below, indicates that under certain conditions this cell can also be an SZ. For the rest of the discussion it is useful to define $\delta(\mathbf{A})$ by

$$\delta(\mathbf{A}) = [r(\mathbf{X}_{min}) + r(\mathbf{A})]. \tag{12}$$

From the inequality (3) it follows that for any $\mathbf{A}$,

$$-1 \leq \delta(\mathbf{A}) < 1, \tag{13}$$

so that, from the definition of $[x]$, $\delta(\mathbf{A})$ can only take on the values, -1, 0, or 1.

Theorem 4 examines the case of the upper left-hand corner cell of the standard parallelogram.

Theorem 4: If $[\mathbf{A}] = [\mathbf{A}_{min}]$, then $[\mathbf{X} + \mathbf{A}] > [\mathbf{X}_{min}] + [\mathbf{A}_{min}]$ if and only if $\delta(\mathbf{A}_{min}) = 1$.

Proof: Clearly, $[\mathbf{X} + \mathbf{A}] \geq [\mathbf{X}_{min} + \mathbf{A}_{min}] = [\mathbf{X}_{min}] + [\mathbf{A}_{min}] + [r(\mathbf{X}_{min}) + r(\mathbf{A}_{min})]$, from which the result follows. QED.

The combined effect of Theorems 1 to 4 is to show that, except for the upper left-hand cell of the standard parallelogram, the lower triangle and the standard parallelogram describe the SZs and non-SZs of rounded formula scores in exactly the same way as they do for consecutive integer scores.

Now it is time to examine the areas of Figure 2 denoted as the super diagonal and the upper triangle. These areas are relevant to the *smallest* values that can be achieved for $[\mathbf{X} + \mathbf{A}]$ for a given value of $[\mathbf{A}]$.

The lower bound that parallels Theorem 1 is weaker and there are various special cases to examine. A basic result is given in Theorem 5.

Theorem 5: If $[\mathbf{A}] = j$, then

$$[\mathbf{X} + \mathbf{A}] \geq [\mathbf{X}_{min}] - 1 + j. \tag{14}$$

Proof: From (7), examine $[\mathbf{X}_{min} + \mathbf{A}]$. Next, (8) shows that

$$[\mathbf{X}_{min} + \mathbf{A}] = [\mathbf{X}_{min}] + [\mathbf{A}] + [r(\mathbf{X}_{min}) + r(\mathbf{A})] = [\mathbf{X}_{min}] + j + \delta(\mathbf{A}). \tag{15}$$

Now apply (13) so that $\delta(\mathbf{A}) \geq -1$, from which (14) follows. QED.

Theorem 5 shows that the upper triangle in Figure 2 only contains SZs for the joint distribution of $[\mathbf{X} + \mathbf{A}]$ and $[\mathbf{A}]$. To complete the picture, examine the super diagonal in Figure 2. Note that the super diagonal includes the cell or score combination defined by $[\mathbf{A}] = [\mathbf{A}_{min}]$ and $[\mathbf{X} + \mathbf{A}] = [\mathbf{X}_{min}] + [\mathbf{A}_{min}] - 1$, even though this cell was not included in Figure 1. For clarity, the super diagonal is defined by the following combinations of scores :

$$[\mathbf{X} + \mathbf{A}] = [\mathbf{X}_{min}] + j - 1 \text{ and } [\mathbf{A}] = j, \text{ for } j = [\mathbf{A}_{min}] \text{ to } [\mathbf{A}_{max}]. \tag{16}$$

The inequality in Theorem 5 can be strengthened when $\mathbf{A}$ is a formula score and $[\mathbf{A}] = \mathbf{A}_{max}$. This is done in Theorem 6.

Theorem 6: If $\mathbf{A}$ is a formula score and $[\mathbf{A}] = \mathbf{A}_{max}$, then

$$[\mathbf{X} + \mathbf{A}] \geq [\mathbf{X}_{min}] + \mathbf{A}_{max}. \tag{17}$$

Proof: From (7), examine $[\mathbf{X}_{min} + \mathbf{A}]$ and then from (8) conclude that,

$$[\mathbf{X}_{min} + \mathbf{A}] = [\mathbf{X}_{min}] + [\mathbf{A}] + [r(\mathbf{X}_{min}) + r(\mathbf{A})] = [\mathbf{X}_{min}] + \mathbf{A}_{max} + \delta(\mathbf{A}).$$

But from Lemma 2b, since $\mathbf{A}$ is a formula score and $[\mathbf{A}] = \mathbf{A}_{max}$, then $\mathbf{A} = \mathbf{A}_{max,}$ an integer so that $r(\mathbf{A}) = 0$ and hence $\delta(\mathbf{A}) = [r(\mathbf{X}_{min}) + 0] = 0$, so that

$$[\mathbf{X} + \mathbf{A}] \geq [\mathbf{X}_{min}] + \mathbf{A}_{max}. \text{ QED.}$$

Theorem 6 shows that the right most cell of the super diagonal is always an SZ for rounded formula scores. The cells of the rest of the diagonal may or may not be SZs. The proof of Theorem 6 shows that $\delta(\mathbf{A}_{max}) = 0$.

Revisiting the proof of Theorem 5, one sees that the lower bound in (14) may be analyzed more carefully. Equation (15) may be written as

$$[\mathbf{X}_{min} + \mathbf{A}] = [\mathbf{X}_{min}] + [\mathbf{A}] + \delta(\mathbf{A}). \tag{18}$$

Equation (18) is the lower bound for $[\mathbf{X} + \mathbf{A}]$ when $[\mathbf{A}]$ is a given value. When $\delta(\mathbf{A}) = -1$, then the lower bound is in the super diagonal. In that case, the corresponding cell in the super diagonal is a non-SZ. If $\delta(\mathbf{A}) = 0$, then the lower bound is below the super diagonal, on the upper boundary of the standard parallelogram, and the corresponding cell above it, in the super diagonal, is an SZ. Finally, Theorem 2 shows that the case of $\delta(\mathbf{A}) = 1$ can be ignored because Theorem 2 shows that cells below the upper boundary of the standard parallelogram are never SZs for formula scores, except for the case described in Theorem 4.

Before addressing the cells of the super diagonal in more detail, let me dispose of the leftmost cell of the super diagonal. Theorem 7 is similar to Theorem 4 in that it concerns a single cell of Figure 2, the one in the super diagonal just above the upper left-hand cell of the standard parallelogram.

Theorem 7: If $[\mathbf{A}] = [\mathbf{A}_{min}]$, then $[\mathbf{X}_{min} + \mathbf{A}_{min}] = [\mathbf{X}_{min}] + [\mathbf{A}_{min}] - 1$ if and only if $\delta(\mathbf{A}_{min}) = -1$.

Proof: From (8) and the definition of $\delta(\mathbf{A})$,

$$[\mathbf{X}_{min} + \mathbf{A}_{min}] = [\mathbf{X}_{min}] + [\mathbf{A}_{min}] + \delta(\mathbf{A}_{min}),$$

from which the result follows. QED.

Theorem 7 shows that the leftmost cell of the super diagonal can be achieved by $[\mathbf{X} + \mathbf{A}]$ if and only if $\delta(\mathbf{A}_{min}) = -1$. Theorem 4 implies that neither that cell nor the one below it can be achieved by $[\mathbf{X} + \mathbf{A}]$ if $\delta(\mathbf{A}_{min}) = 1$.

The cells of the super diagonal can be achieved by $[\mathbf{X} + \mathbf{A}]$ and $[\mathbf{A}]$ if and only if a value of $\mathbf{A}$ exists such that $[\mathbf{A}] = j$ and $[\mathbf{X} + \mathbf{A}] = [\mathbf{X}_{min}] + j - 1$. This can never happen for $j = \mathbf{A}_{max}$ for formula scores, and Theorem 7 gives the only condition when it can happen for $j = [\mathbf{A}_{min}]$ for formula scores. So now suppose that $\mathbf{A}$ is a formula score and that

$$[\mathbf{A}_{min}] < j < \mathbf{A}_{max}. \tag{19}$$

The problem of the super diagonal reduces to the following question. If j satisfies (19), when does a value of $\mathbf{A}$ exist such that $[\mathbf{A}] = j$ and $\delta(\mathbf{A}) = -1$? If $\mathbf{A}$ can be chosen so that $[\mathbf{A}] = j$ and $\delta(\mathbf{A}) = -1$, then the corresponding score on the super diagonal can be achieved and is a non-SZ. If $\delta(\mathbf{A}) > -1$ for any $\mathbf{A}$ such that $[\mathbf{A}] = j$, then the corresponding score on the super diagonal cannot be achieved and is an SZ.

Theorem 8 gives the condition under which $\delta(\mathbf{A}) = -1$.

Theorem 8: $\delta(\mathbf{A}) = -1$ if and only if $-1 \le r(\mathbf{X}_{min}) + r(\mathbf{A}) < -0.5$.

Proof: By definition, $\delta(\mathbf{A}) = [r(\mathbf{X}_{min}) + r(\mathbf{A})]$. Clearly, $r(\mathbf{X}_{min}) + r(\mathbf{A})$ can round to -1 if and only if $-1 \le r(\mathbf{X}_{min}) + r(\mathbf{A}) < -0.5$. QED.

It is easy to show that the condition, $-1 \le r(\mathbf{X}_{min}) + r(\mathbf{A}) < -0.5$, in Theorem 8, and the inequality (3) combine to imply that both $r(\mathbf{X}_{min})$ and $r(\mathbf{A})$ must be strictly less than 0. Theorem 9 finishes off the remaining cells of the super diagonal. It shows that either all of the cells are possible score combinations, except for the rightmost cell, or none of them are.

Theorem 9: If $\mathbf{A}$ is a formula score with $[\mathbf{A}] = j < \mathbf{A}_{max}$ and $\delta(\mathbf{A}_{min}) = -1$, then $[\mathbf{X} + \mathbf{A}]$ can achieve any score of the form $[\mathbf{X}_{min}] + j - 1$.

Proof: From Theorem 8, $\delta(\mathbf{A}_{min}) = -1$ if and only if $-1 \le r(\mathbf{X}_{min}) + r(\mathbf{A}_{min}) < -0.5$, so that $r(\mathbf{A}_{min}) < 0$. Hence,

$$\mathbf{A}_{min} < [\mathbf{A}_{min}]. \tag{20}$$

Now suppose $\mathbf{A}$ is such that $[\mathbf{A}] = j < \mathbf{A}_{max}$. Is it possible to find an $\mathbf{A}$ such that $r(\mathbf{X}_{min}) + r(\mathbf{A}) < -0.5$, as is required in Theorem 8? Clearly by hypothesis $\mathbf{A}_{min}$ satisfies this for $[\mathbf{A}] = [\mathbf{A}_{min}]$. However, a corresponding value of $\mathbf{A}$ for $[\mathbf{A}] = [\mathbf{A}_{min}] + 1$ can be found by reducing the number of wrong responses appropriately and replacing them with omitted or correct responses, thus increasing $\mathbf{A}_{min}$ by 1 point. This approach steps through all of the values of $[\mathbf{A}] = j < \mathbf{A}_{max}$. QED.

Together, Theorems 7 and 9 imply that if $\delta(\mathbf{A}_{min}) = -1$, then all of the super diagonal in Figure 2 is achievable by $[\mathbf{X} + \mathbf{A}]$ and $[\mathbf{A}]$ except for the rightmost cell; that, by Theorem 6, is always an SZ. If $\delta(\mathbf{A}_{min}) > -1$ then the super diagonal only contains SZs.

The Final Result for Rounded Formula Scores

The three possibilities are as follows:

1. If $\delta(\mathbf{A}_{min}) = 0$, then the standard parallelogram describes all of the non-SZs for the joint distribution and all of the other cells are SZs.

2. If $\delta(\mathbf{A}_{min}) = 1$, then the standard parallelogram describes all of the non-SZs for the joint distribution except for the upper leftmost cell, which is also an SZ. The remaining cells are all SZs.

3. If $\delta(\mathbf{A}_{min}) = -1$, then the standard parallelogram plus the super diagonal minus its rightmost cell describe the non-SZs for the distribution. All the other cells are SZs.

The examples of section 5 illustrate these three possibilities.

From this discussion, the value of $\delta(\mathbf{A}_{min})$ determines where the SZs and non-SZs of the joint distribution of $[\mathbf{X} + \mathbf{A}]$ and $[\mathbf{A}]$ are located. Clearly, the standard parallelogram plays a major role, as does the super diagonal that lies directly above it. Computer programs that are used to locate the SZs for such joint distributions need to know the value of

$$\delta(\mathbf{A}_{min}) = [r(\mathbf{X}_{min}) + r(\mathbf{A}_{min})]. \tag{21}$$

Whether this should be done by having the value of $[r(\mathbf{X}_{min}) + r(\mathbf{A}_{min})]$ supplied by the user or by the program computing it from other information the user provides is a judgment that programmers will need to make.

5. Three Simple Examples of Rounded Formula Scores

Example 1: (An example of $\delta(\mathbf{A}_{min}) = 1$.) **X** and **A** both have three five-option items with R – (1/4)W being the formula score:

$$\mathbf{X}_{min} = -3/4,\ [\mathbf{X}_{min}] = -1,\ r(\mathbf{X}_{min}) = 1/4 > 0,$$

$$\mathbf{A}_{min} = -3/4,\ [\mathbf{A}_{min}] = -1,\ r(\mathbf{A}_{min}) = 1/4 > 0,$$

$$\delta(\mathbf{A}_{min}) = [1/4 + 1/4] = [0.5] = 1.$$

In this example, $[\mathbf{X}_{min} + \mathbf{A}_{min}] = [-6/4] = -1$, while $[\mathbf{X}_{min}] + [\mathbf{A}_{min}] = -2$, so that it is impossible for $[\mathbf{A}] = [\mathbf{A}_{min}] = -1$ and $[\mathbf{X} + \mathbf{A}] = [\mathbf{X}_{min}] + [\mathbf{A}_{min}] = -2$. If 0 denotes an SZ and 1 a non-SZ, then Table 1 corresponds to Figure 2 for this example. To that end, the standard parallelogram and the extra SZ within it is shaded differently. Note that this example can be generalized to any **X** and **A** with 4n + 3 and 4m + 3 five-option MC items, respectively.

Table 1

The Location of SZs for Example 1

	[**A**]				
[**X**+**A**]	-1	0	1	2	3
-2	0	0	0	0	0
-1	1	1	0	0	0
0	1	1	1	0	0
1	1	1	1	1	0
2	1	1	1	1	1
3	0	1	1	1	1
4	0	0	1	1	1
5	0	0	0	1	1
6	0	0	0	0	1

Example 2: (An example of $\delta(\mathbf{A}_{min}) = -1$.) **X** and **A** both have two five-option items with R – (1/4)W being the formula score:

$$\mathbf{X}_{min} = -2/4,\ [\mathbf{X}_{min}] = 0,\ r(\mathbf{X}_{min}) = -0.5 < 0,$$

$$\mathbf{A}_{min} = -2/4,\ [\mathbf{A}_{min}] = 0,\ r(\mathbf{A}_{min}) = -0.5 < 0,$$

$$\delta(\mathbf{A}_{min}) = [-0.5 - 0.5] = [-1] = -1.$$

In this example, $[\mathbf{X}_{min} + \mathbf{A}_{min}] = [-1] = -1$, while $[\mathbf{X}_{min}] + [\mathbf{A}_{min}] = 0$, so that it is possible for $[\mathbf{A}] = 0$ and $[\mathbf{X}_{min} + \mathbf{A}_{min}] = -1$. Table 2 corresponds to Figure 2 for this example. To that end, the super diagonal and the standard parallelogram are shaded differently. Note that this example can be generalized to any **X** and **A** with 4n + 2 and 4m + 2 five-option MC items, respectively.

Table 2

The Location of SZs for Example 2

	[**A**]		
[**X**+**A**]	0	1	2
-1	1	0	0
0	1	1	0
1	1	1	0
2	1	1	1
3	0	1	1

4	0	0	1

Example 3: (An example of $\delta(\mathbf{A}_{min}) = 0$.) **X** has two five-option items and **A** has three five-option items with R – (1/4)W being the formula score:

$$\mathbf{X}_{min} = -2/4,\ [\mathbf{X}_{min}] = 0,\ r(\mathbf{X}_{min}) = -0.5 < 0,$$

$$\mathbf{A}_{min} = -3/4,\ [\mathbf{A}_{min}] = -1,\ r(\mathbf{A}_{min}) = 1/4 > 0,$$

$$\delta(\mathbf{A}_{min}) = [-0.5 + 0.25] = [-0.25] = 0.$$

In this example $[\mathbf{X}_{min} + \mathbf{A}_{min}] = [-5/4] = -1$, and $[\mathbf{X}_{min}] + [\mathbf{A}_{min}] = 0 - 1 = -1$, so that it is possible for $[\mathbf{A}] = -1$ and $[\mathbf{X}_{min} + \mathbf{A}_{min}] = -1$, as well. Table 3 corresponds to Figure 2 for this example. the standard parallelogram is shaded. Note that this example can be generalized to any **X** with 4n + 2 and **A** with 4m + 3 five-option MC items.

Table 3

The Location of SZs for Example 3

			[**A**]		
[**X**+**A**]	-1	0	1	2	3
-1	1	0	0	0	0
0	1	1	0	0	0
1	1	1	1	0	0
2	0	1	1	1	0
3	0	0	1	1	1
4	0	0	0	1	1
5	0	0	0	0	1

6. Implications for Degrees of Freedom for Chi-Square Tests

In fitting loglinear models to bivariate score distributions as proposed, for example in Holland and Thayer (2000), the issue of the effect of SZs on the distributions of chi-square statistics naturally arises. The effect on the nominal degrees of freedom is easy to describe. In general, the effect of SZs is to reduce the nominal degrees of freedom by the number of SZs. For the cases that interest us here, this can be quite a substantial reduction.

As an example, suppose the scores are all number-right and that there are J items in **X** and L items in **A**. Thus, the cross tabulation of **X** + **A** with **A** has a total of (J + L + 1)(L + 1) cells in it. The upper triangle in Figure 1 then contains

$$L + (L - 1) + (L - 2) + \ldots + 1 = \tfrac{1}{2} L(L + 1)$$

cells, and the lower triangle contains

$$1 + 2 + \ldots + L = \tfrac{1}{2} L(L + 1)$$

cells as well. Hence, the total number of SZs is L(L + 1). Reducing (J + L + 1)(L + 1) cells by L(L + 1) is easily shown to yield (J + 1)(L + 1), which is just the number of possible score combinations for ($\mathbf{X}$, $\mathbf{A}$). The nominal degrees of freedom are

$$\text{Nominal DF} = (J + 1)(L + 1) - 1 - s, \tag{22}$$

where s is the number of free parameters in the estimated loglinear model.

When the scores are rounded formula scores J + 1 is replaced by

$$\mathbf{X}_{max} - [\mathbf{X}_{min}] + 1$$

and L + 1 by

$$\mathbf{A}_{max} - [\mathbf{A}_{min}] + 1.$$

In this case the analogue of (J + 1)(L + 1) – 1 – s is

$$\text{SP DF} = (\mathbf{X}_{max} - [\mathbf{X}_{min}] + 1)(\mathbf{A}_{max} - [\mathbf{A}_{min}] + 1) - 1 - s. \tag{23}$$

However, in this case the value of $\delta(\mathbf{A}_{min})$ needs to be taken into account as well. There may be additional or fewer SZs, as indicated in the final result of section 3.

When $\delta(\mathbf{A}_{min}) = 0$, there are no additional SZs, and SP DF in (23) is the nominal degrees of freedom.

When $\delta(\mathbf{A}_{min}) = 1$, there is one additional SZ so that the nominal degrees of freedom are reduced by one, SP DF – 1.

Finally, when $\delta(\mathbf{A}_{min}) = -1$, there are an additional $\mathbf{A}_{max} - [\mathbf{A}_{min}]$ non-SZs on the super diagonal of Figure 2 so that the nominal degrees of freedom are then increased to

$$\text{SP DF} + \mathbf{A}_{max} - [\mathbf{A}_{min}].$$

While it is fairly easy to describe how the nominal degrees of freedom change as a result of the SZs in the cases described in this report, in real applications the nominal degrees of freedom are, themselves, of limited use. This is due to the large degree of sparseness that obtains in most observed bivariate score distributions. This sparseness of data usually invalidates a direct interpretation of the nominal degrees of freedom as *the* degrees of freedom applicable to chi-square tests. However, differences in the nominal degrees of freedom for two-nested loglinear models are often valid, as the degrees of freedom applicable to the difference in the corresponding likelihood ratio chi-square tests for the two nested models. What really matters in these tests of nested models is the difference in their number of parameters, that is, the *s* in (22) and (23), which does not require a user to know the nominal degrees of freedom in (22) or (23).

7. Summary

SZs arise in the bivariate distribution of a total score and a part score of the total test. When fitting models to such bivariate distributions, as is routinely done in various equating situations, the models should not put positive probability into any SZ. Therefore, it is important to be able to locate where the SZs are because the data can not distinguish between an SZ and a zero frequency that is due to sampling variability and low population proportions. In the case of number-right scoring, it is fairly easy to locate the SZs.. Example 3 illustrates this case. In the case where the scores for both the total test and the part test are rounded formula scores, there are three cases that can arise. The quantity $\delta(\mathbf{A}_{\min})$, defined in (21) can be used to identify which case arises in a particular situation. These three cases are illustrated in Examples 1, 2, and 3.

References

Bishop, Y. M. M., Fienberg, S. E, & Holland, P. W. (1975). *Discrete multivariate analysis: Theory and practice*. Cambridge, M**A**: MIT Press.

von Davier, **A**. **A**., Holland, P. W., & Thayer, D. T. (2004). *The kernel method of test equating*. New York: Springer.

Holland, P. W., & Thayer, D. T. (2000). Univariate and bivariate loglinear models for discrete test score distributions. *Journal of Educational and Behavioral Statistics*, *25,* 133–183.